NOTE ON RADIAL VIBRATION OF AN AEOLOTROPIC CYLINDRICAL SHELL IN A MAGNETIC FIELD

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ABSTRACT. The radial vibration in an aelotropic cylindrical shell has been investigated by solving the equations of elasticity, taking into account the effect of a magnetic field and the electromagnetic equations of Maxwell and finally the frequency equation is obtained.

INTRODUCTION

The problems of magnetoelastic vibration are considered to be important in view of increasing investigation on radiation of electromagnetic energy into the vacuum adjacent to magnetoelastic bodies. Such problems have been discussed in a series of papers by Kaliski. Kaliski (1963*a*, *b*) has dealt with the resonance vibration of perfectly conducting plate considering the influence of magnetoelastic radiation in vacuum and magnetoelastic resonance of perfectly conducting cylinder in the axial magnetic field respectively. Sinha (1965) has studied the torsional disturbances in a elastic cylinder in a magnetic field. In a recent paper, Giri (1965) has discussed the magnetoelastic disturbances in a transversely isotropic elastic cylinder in a magnetic field. As a sequel to these papers, the present note is an attempt to investigate the radial vibration in an aeolotropic cylindrical shell in magnetic field.

PROBLEM, FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS

Let us consider an aelotropic perfectly conducting cylindrical shell of inner and outer radii r_1 and r_2 respectively. The space outside the shell is considered to be vacuum. The boundary of the shell is supposed to be mechanically stress free. Initially there exists in the shell, an axial magnetic field of intensity H. As the main object is to determine the vibration set by the magnetic field into the medium, we shall consider the expression connecting the components of stress and strain. The constitutive relations for aelotropic bodies in cylindrical coordinates as in Love (1952) are given as

$$\begin{aligned} \sigma_{rr} &= C_{11}S_{rr}+C_{12}S_{\theta\theta}+C_{13}S_{zz} \\ \sigma_{\theta\theta} &= C_{21}S_{rr}+C_{22}S_{\theta\theta}+C_{23}S_{zz} \\ \sigma_{zz} &= C_{31}S_{rr}+C_{32}S_{\theta\theta}+C_{22}S_{zz}. \end{aligned} \quad .. \quad (2.1)$$

where σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} and S_{rr}, $S_{\theta\theta}$, S_{zz} are the components of stress and strain respectively, C_s the material constants.

The cylindrical coordinates (r, θ, z) are considered to be the coordinates of reference.

The equations of magnetoelasticity for a perfect conductor with unit permeability, as deduced by Kaliski (1963a, b)

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \frac{\partial \sigma_{\gamma\gamma}}{\partial r} + \frac{\sigma_{\gamma\gamma} - \sigma_{\theta\theta}}{r} + \frac{1}{4\pi} [\text{rot rot}(\vec{u} \times \vec{H})] \times \vec{H} \quad \text{.. (2.2)}$$

$$\vec{E} = -\frac{1}{C}\left(\frac{\partial \bar{u}}{\partial t} \times \vec{H}\right), \qquad \vec{h} = \text{rot}\,(\vec{u} \times \vec{H}) \quad \text{.. (2.3)}$$

where $\vec{u}$ is mechanical displacement vector $\vec{E}$ the electric intensity vector and $\vec{h}$ is the perturbation of the magnetic intensity vector.

The equations electromagnetic field in vacuum are given

$$\left(\Delta^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}^* = 0 \quad \text{.. (2.4)}$$

$$\left(\Delta^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{h}^* = 0 \quad \text{.. (2.5)}$$

$$\text{rot}\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{h}^*}{\partial t} \quad \text{.. (2.6)}$$

$$\text{rot}\,\vec{h} = \frac{1}{c}\frac{\partial \vec{E}^*}{\partial t} \quad \text{.. (2.7)}$$

where $\vec{E}^*$, $\vec{h}^*$ denote the values of quantities $\vec{E}$, $\vec{h}$, respectively in vacuum.

For the radial vibration we assume

$$u_\theta = u_z = 0, \qquad u_\gamma = Ue^{i\omega t} \quad \text{.. (2.8)}$$

and the other corresponding quantities as

$$\left.\begin{array}{ll} h^* = h_\theta^* = 0, & h_z = h = Ve^{i\omega t} \\ H_\gamma = H_\theta = 0, & H_z = H_1 \\ E_\gamma^* = E_z^* = 0, & E_\theta = E^* = We^{i\omega t} \end{array}\right\} \quad \text{.. (2.9)}$$

where U, V, W, are the function or r alone.

Equation (1) with the help of (2.2) and (2.8) becomes

$$\frac{d^2U}{dr^2} + \frac{1-\alpha}{r} + \left(n^2\omega^2 - \frac{\lambda^2}{r^2}\right)U = 0 \qquad .. \quad (2.10)$$

where

$$\alpha = -\frac{(C_{12}-C_{21})}{C_{11}+\frac{H_1 2}{4\pi}}$$

$$n^2 = \frac{\rho}{C_{11}+\frac{H_1 2}{4\pi}}$$

$$\lambda^2 = \frac{C_{22}}{C_{11}+\frac{H_1 2}{4\pi}}$$

Equations (2.6) and (2.7) are modified into

$$E = \frac{H_1}{c}\frac{\partial U}{\partial \gamma} = \frac{i\omega}{c} H_1 U e^{i\omega t}$$

$$h = \frac{H_1}{r}\frac{\partial}{\partial r}(ru) = H_1\left(\frac{dU}{dr} + \frac{U}{r}\right)e^{i\omega t} \qquad .. \quad (2.11)$$

From (2.5) and (2.6), we get

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} + \frac{\omega^2}{c^2}V = 0 \qquad .. \quad (2.12)$$

$$W = i\frac{c}{\omega}\frac{dV}{dr} \qquad .. \quad (2.13)$$

The boundary conditions on the surface are

$$\begin{aligned} \sigma_{rr}+T_{rr} &= T_{rr}^* && \text{on } r = r_1 \\ \sigma_{rr}+T_{rr} &= T_{rr}^* && \text{on } r = r_2 \\ E &= E^* && \text{on } r = r_2 \\ E &= E^* && \text{on } r = r_1 \qquad .. \quad (2.14) \end{aligned}$$

where T_{rr}, T_{rr}^* are Maxwell tensors in the shell and vacuum respectively. The first two boundary conditions show that the total stress-electro-magnetic stress and mechanical stress—is continuous on the boundary surfaces. The other conditions represent the conditions of continuity of the tangential component of the electric field on the surfaces.

The Maxwell tensors T_{rr}, T_{rr}^* can be expressed as

$$T_{rr} = -\frac{1}{4\pi} H_1 h = \frac{H_1^2}{4\pi}\left(\frac{du}{dr}+\frac{u}{r}\right) e^{i\omega t}$$

$$T_{rr}^* = -\frac{H_1}{4\pi} h^* = \frac{H_1}{4\pi} V e^{i\omega t} \qquad .. \quad (2.15)$$

while the elastic stress tensor is

$$\sigma_{rr} = \left(C_{11}\frac{du}{dr} + C_{12}\frac{u}{r}\right)e^{i\omega t} \qquad .. \quad (216.)$$

METHOD OF SOLUTION

The solution of (2.10) is

$$U = r^{\alpha/2}[A J_\mu(n\omega r) + B\, Y_\mu(n\omega r)] \qquad .. \quad (3.1)$$

J_μ, Y_μ are Bessel functions of 1st and 2nd kind of order μ.

where $\mu^2 = \dfrac{4\lambda^2-\alpha^2}{4}$ and A, B are constants.

Making the use of recurrence formula, σ_{rr}, T_{rr} may be calculated by the help of (3.1)

$$\sigma_{rr} = r^{\alpha/2-1}\left[A\left\{\left(\frac{\alpha}{2}C_{11}+\mu C_{11}+C_{12}\right)J_\mu(n\omega r) - C_{11}(n\omega t)J_{\mu+1}(n\omega r)\right\}\right.$$

$$\left. + B\left\{\left(\frac{\alpha}{2}C_{11}+\mu C_{11}+C_{12}\right)Y_\mu(n\omega r) - C_{11}(n\omega r)Y_{\mu+1}(n\omega r)\right\}\right] e^{i\omega t}. \qquad .. \quad (3.2)$$

$$T_{rr} = \frac{H_1^2}{4\pi} r^{\alpha-1}\left[A\left\{\left(\frac{\alpha}{2}+\mu+1\right)J_\mu(n\omega r) - (n\omega r)J_{\mu+1}(n\omega r)\right\}\right.$$

$$\left. + B\left\{\left(\frac{\alpha}{2}+\mu+1\right)J_\mu(n\omega r) - (n\omega r)Y_{\mu+1}(n\omega r)\right\}\right] e^{i\omega t}. \qquad .. \quad (3.3)$$

From (3.1) and (2.11), we get

$$E = \frac{i\omega}{c} H_1 r^{\alpha/2}[A J_\mu(n\omega r) + B Y_\mu(n\omega r)]e^{i\omega t} \qquad .. \quad (3.4)$$

Solution of (2.12) with appropriate conditions is obtained as

$$V = CY_0\left(\frac{\omega r}{c}\right) \qquad \text{when} \quad r > r_2$$

$$= DJ_0\left(\frac{\omega r}{c}\right) \qquad \text{when} \quad r < r_1 \qquad .. \quad (3.5)$$

where Y_0 and J_0 are Bessel function of first and of second kind and of order zero.

$$T_{rr}^* = \frac{H_1}{4\pi} CY_0\left(\frac{\omega r}{c}\right) e^{i\omega t} \quad \text{on} \quad r \geqslant r_2$$

From (2.15), we get .. (3.6)

$$T_{rr}^* = \frac{H_1}{4\pi} DJ_0\left(\frac{\omega r}{c}\right) e^{i\omega t} \quad \text{on} \quad r \leqslant r_1$$

From (3.5) and (2.13), we get W. Hence (2.9) gives

$$E^* = iCY_1\left(\frac{\omega r}{c}\right) e^{i\omega t} \qquad \text{on} \quad r \geqslant r_2$$

$$E^* = iDJ_1\left(\frac{\omega r}{c}\right) e^{i\omega t} \qquad \text{on} \quad r \leqslant r_1 \qquad .. \quad (3.7)$$

Boundary conditions (2.14) yield

$$[A\{\theta_1 J_\mu(n\omega r) - \theta_2 r_1 J_{\mu+1}(n\omega r)\} + B\{\theta_1 Y_\mu(n\omega r_1) - \theta_2 r_1 Y_{\mu+1}(n\omega r_1)\}]$$

$$= \frac{H_1}{4\pi}\, r_1^{(4-\alpha/2)} DJ_0\left(\frac{\omega r_1}{c}\right) \qquad .. \quad (3.8)$$

where

$$\theta_1 = \left[\frac{\alpha}{2}\left(C_{11} + \frac{H_1^2}{4\pi}\right) + \mu\left(C_{11} + \frac{H_1^2}{4\pi}\right) + \left(C_{12} + \frac{H_1^2}{4\pi}\right)\right]$$

$$\theta_2 = n\omega\left(C_{11} + \frac{H_1^2}{4\pi}\right)$$

$$[A\{\theta_1 J_\mu(n\omega r_2) - \theta_2 r_2 J_{\mu+1}(n\omega r_2)\} + \{\theta_1 J_\mu(n\omega r_2) - \theta_2 r_2 J_{\mu+1}(n\omega r_2)\}]$$

$$= \frac{H_1}{4\pi}\, r^{(1-\alpha/2)}\, CY_0\left(\frac{\omega r_2}{c}\right) \qquad .. \quad (3.9)$$

$$AJ_\mu(n\omega r_2)+BY_\mu(n\omega r_1) = \frac{C}{\omega H_1 r_2^{\alpha/2}}\, CY_1\left(\frac{\omega r_2}{c}\right) \quad .. \quad (3.10)$$

$$AJ_\mu(n\omega r_1)+BY_\mu(n\omega r_o) = \frac{C}{\omega H_1 r^{\alpha/2}}\, DJ_1\left(\frac{\omega r_1}{c}\right) \quad .. \quad (3.11)$$

Eliminating A, B, C, D from (3.8), (3.9), (3.10), (3.11) we obtain the frequency equation as

$$\frac{\left[\left\{\frac{H_1}{4\pi}\, r^{(1-\alpha/2}J_0\left(\frac{\omega r_1}{c}\right)\right\}\{Y_\mu(n\omega r_1)\}-\right.}{\left[\{\theta_1 J_\mu(n\omega r_1)-\theta_2 r_1 J_{\mu+1+1}(n\omega r_1)\}\left\{\frac{C}{\omega H_1 r_1^{\alpha/2}}\, J_1\left(\frac{\omega r_1}{c}\right)\right\}\right]-}$$

$$\frac{-\left[\left\{\frac{C}{\omega H_1 r_1^{\alpha/2}} J_1\left(\frac{\omega r_1}{c}\right)\right\}\{\theta_1 Y_\mu(n\omega r_1)-\theta_2 r_1 Y_{\mu+1}(n\omega r_1)\}\right]}{-\left[\left\{\frac{H_1}{4\pi}\, r_1^{(1-\alpha/2)}J_0\left(\frac{\omega r_1}{c}\right)\right\}\{J_\mu(n\omega r_1)\}\right]}$$

$$=\frac{\left[\left\{\frac{H_1}{4\pi}\, r_2^{(1-\alpha 2)}Y_0\left(\frac{\omega r_2}{c}\right)\right\}\{Y_\mu(n\omega r_2)\}\right]-}{\left[\left\{\frac{C}{\omega H_1 r_2^{\alpha/2}}\, Y_1\left(\frac{\omega r_2}{c}\right)\right\}\{\theta_1 J_\mu(n\omega r_2)-\theta_2 r_2 J_{\mu+1}(n\omega r_2)\}\right]-}$$

$$\frac{-\left[\{\theta_1 Y_\mu(n\omega r_2)-\theta_2 r_2 Y_{\mu+1}(n\omega r_2)\}\left\{\frac{C}{\omega H_1 r_2^{\alpha/2}}\, Y_0\left(\frac{\omega r^2}{c}\right)\right\}\right]}{-\left[\{J_\mu(n\omega r_2)\}\left\{\frac{H_1}{4\pi}\, r_2^{(1-\alpha/2)}Y_0\left(\frac{\omega r_2}{c}\right)\right\}\right]}$$

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